

A proposed Faraday cage experiment

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The E field sensed by a moving charged particle inside a Faraday cage is obtained by applying the Liénard-Wiechert retardation equations in the frame of reference of the particle. The solution indicates that, depending on its sign, a charged particle is accelerated in either the same or the opposite direction of its velocity vector by an E field of order v/c times the E field at the external surface of the cage. The field is large enough for laboratory evaluation.

I. INTRODUCTION

Despite refined and sensitive attempts², an E field within a Faraday cage has never been detected with stationary charged particles. In order to investigate the solution for moving particles, the retardation solution for two charged particles of the same sign and magnitude at the locations $+z$ and $-z$ will be needed. Both particles are moving in the z direction with the velocity v_z .

All of the calculations in this paper are shown in more detail in the supplemental online material (SOM) at www.s-4.com/faraday.

With L_{1s} being the location of the source particle and L_{1f} being the location of the field point, the coordinates are

$$\begin{aligned} L_{1s} &= [[0, r \sin \theta, r \cos \theta + t_s v_z], t_s] \\ L_{1f} &= [[0, 0, dz], dt]. \end{aligned} \quad (1)$$

t_s is the time at the source. The difference of the two locations is

$$L_{1s} - L_{1f} = [[0, r \sin \theta, r \cos \theta + t_s v_z - dz], t_s - dt].$$

With \mathbf{r} being the space part of the difference and t being the time part of the difference, the light cone condition is $\mathbf{r} \cdot \mathbf{r} - c^2 t^2 = 0$. The solution has two roots. Choosing the one that is applicable to the problem and expanding it in a series,

$$t_s = dt - dz v_z/c^2 + r v_z \cos \theta/c^2 - dt v_z \cos \theta/c + dz \cos \theta/c - r/c.$$

There are higher powers of v_z in the solution, but only the first power will be carried in these calculations. After substituting for t_s in Eqs. 1 the retarded locations become

$$\begin{aligned} L_{1s} &= [[0, r \sin \theta, dt v_z + r \cos \theta + dz v_z \cos \theta/c \\ &\quad - r v_z/c], dt - dz v_z/c^2 + r v_z \cos \theta/c^2 \\ &\quad - dt v_z \cos \theta/c + dz \cos \theta/c - r/c] \end{aligned} \quad (2)$$

$$L_{1f} = [[0, 0, dz], dt].$$

Applying the Lorentz transform to these locations provides the coordinates in the second frame of reference. The transform velocity is $[0, 0, v_z]$.

$$L_{2s} = [[0, r \sin \theta, r \cos \theta], dt - dz v_z/c^2 - dt v_z \cos \theta/c + dz \cos \theta/c - r/c]$$

$$L_{2f} = [[0, 0, dz - dt v_z], dt - dz v_z/c^2].$$

With \mathbf{r}_2 being the space part of $L_{2s} - L_{2f}$, $(\mathbf{r}_2 \cdot \mathbf{r}_2)^{1/2}$ is

$$r_2 = r + dt v_z \cos \theta - dz \cos \theta \quad (3)$$

The particle is at rest in the second frame of reference. The vector potential is zero and the scalar potential is $q/(4\pi\epsilon_0 r_2)$, or

$$\psi = \frac{q}{4\pi\epsilon_0} (-dt v_z \cos \theta/r^2 + dz \cos \theta/r^2 + 1/r).$$

All calculations are in series form. The 4-potential transforms in the same way as the coordinates. In the SI system of units the vector potential must be multiplied by c^2 before the transform, then be divided by c^2 after the transform. After transforming the potential back to the frame of reference of the field point with the velocity $[0, 0, -v_z]$ the $-\partial\mathbf{A}/\partial t$ contribution to the E field is zero. The contribution of the $-\nabla\psi$ term is

$$E_z = -\frac{q}{4\pi\epsilon_0} \cos \theta/r^2.$$

The solution for the particle at $+z$ is obtained by substituting $\theta = 0$, and the solution for the one at $-z$ is given by the solution for $\theta = \pi$. The sum of the two solutions is

$$E_z = 0. \quad (4)$$

Applying the Liénard-Wiechert (LW) retardation equations³ to the space part of the quantity $L_{1s} - L_{1f}$ in Eqs. 2 provides the solution

$$E_z = \frac{q}{4\pi\epsilon_0} [-v_z/(r^2 c) + 2v_z \cos^2 \theta/(r^2 c) - \cos \theta/r^2].$$

As before, the solution for the particle at $+z$ is obtained by substituting $\theta = 0$, and the solution for the one at $-z$

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is given by the solution for $\theta = \pi$. The sum of the two solutions is

$$E_z = \frac{q}{4\pi\epsilon_0} 2v_z/(r^2c). \quad (5)$$

Given that the calculations leading to Eq. 4 are one of the ways of deriving the LW equations, it may be surprising that the two solutions are different.

For each choice of the values for dt and dz in Eqs. 2 there is a location for the retarded particle that places it on the light cone. dt and dz are both independent variables. The retardation problem can be worked in another, albeit unconventional, way. The other way consists of holding the position of the source fixed and solving the equations for the value of the dependent variable dt as a function of independent variable dz . The solution is obtained by first setting dz and dt to zero in Eqs. 2 then solving the light cone equation a second time for the solution when they are not zero. After selecting the appropriate root and expanding the solution in a series,

$$dt = dz v_z \sin^2 \theta/c^2 - dz \cos \theta/c. \quad (6)$$

Obtaining the solution for r_2 in the same way as for Eq. 3,

$$r_2 = r - dz \cos \theta - dz v_z \cos^2 \theta/c.$$

The basis of the discrepancy between Eqs. 4 and 5 is now clear. The $dz v_z \cos^2 \theta/c$ term is missing in the solution shown in Eq. 3. The term is velocity-dependent and it has the same sign for both particles.

The radius vector is always positive when applying the LW equations, so the substitution $\mathbf{r} \rightarrow -\mathbf{r}$ does not occur when comparing the solutions for particles at $+z$ and $-z$. The easiest way of handling the sign complications of this problem is simply to rely on the LW equations. The basis of the complications is that a sign inversion is not equivalent to a rotation of 180 degrees. (The substitution $\mathbf{r} \rightarrow -\mathbf{r}$ would nevertheless be valid if used consistently.)

After completing the solution as for Eq. 4, the E_z field for both particles is

$$E_z = -\frac{q}{4\pi\epsilon_0} 2v_z/(r^2c).$$

The solution is same as for the LW solution in Eq. 5, except that the sign is inverted. The dual solution for the case where dt is the independent variable is shown in the SOM.

The cross term $dt dz$ is one of the terms of the second derivative. It cannot be consistently carried in the first frame of reference without also carrying terms such as $dz_1 dz_2$. Space and time are orthogonal in the first frame of reference, but they are not orthogonal for light cone relationships. Because dz and dt are not orthogonal in the second frame of reference, constraining dt in Eq. 6 causes a second derivative term to behave like a first derivative. The null result in Eq. 4 assumes that

only the first derivatives in the first frame of reference are important.

Velocities do not have an absolute significance, so if there is an E field inside a moving charged sphere then a moving charged particle within a stationary sphere should be accelerated by an E field.

According to the relativity principle, there is no requirement that the coordinates be first-known in the laboratory frame of reference. It is rather that it should make no difference at all in which frame of reference we begin a calculation, although the solution does eventually have to be expressed in terms of coordinates that are first-known in the laboratory system.

The Thomas precession¹ is of order a^1v^1 , and it is the only known relativistic effect that is not quadratic in velocity. In the following calculations the acceleration is parallel to the velocity vector, causing the Thomas precession to vanish. To first order only, the solution obtained may therefore usable as is in the laboratory frame of reference.

It has been determined that the neglect of the odd derivatives in the following calculations causes the magnitude of the effect to be larger than the solution shown. More refined calculations are in progress, and later versions of this manuscript may be available at www.s-4.com/faraday. It should be possible to perform the calculations with coordinates that are first-known in the laboratory frame of reference. The solution obtained in that way would be more conventional, although less appealing in some ways.

II. THE SPHERICAL FARADAY CAGE

The solution for the Liénard–Wiechert³ retardation equations is shown in detail in the supplemental online material (SOM) at www.s-4.com/faraday. The calculations are for both the first and second derivatives of the potentials at the center of the sphere, as evaluated in the frame of reference of the particle. All of the second derivatives are zero. The only first derivative is

$$E_z = -\frac{2}{3}E v_z/c, \quad (7)$$

where v_z is the velocity of the particle in the z direction in the laboratory frame of reference and E is the field at the external surface of the sphere. The sphere is at rest. No relativistic corrections have been applied to the solution. Furthermore, there are v_z^2 and v_z^3 terms in the frame of reference of the particle that are not carried here, but some solutions are shown in the SOM.

The E field at the outer surface of the sphere is $E = q/(4\pi\epsilon_0 r_0^2)$, and the voltage at the surface, relative to a point at infinity, is $V = q/(4\pi\epsilon_0 r_0)$. Then $E = V/r_0$ and Eq. 7 becomes

$$E_z = -\frac{2}{3}V v_z/(r_0 c). \quad (8)$$

The E field accelerates a charged particle, and it moves by the amount $\frac{1}{2}at^2$ in the frame of reference of the particle during a traverse. In the laboratory frame of reference the velocity acquired by the particle due to the equation $v = at$ is small in relation to v_z when the voltage on the sphere is low, so the time required when traversing the distance L is approximately L/v_z , and the accumulated distance in the frame of reference of the particle is $dz = \frac{1}{2}a(L/v_z)^2$. The force on an electron is $q_e E$ and the acceleration is $a = q_e E/m_e$. dz becomes

$$dz = -\frac{1}{2}(L/v_z)^2 E q_e/m_e.$$

After substituting for E from Eq. 8

$$dz = -L^2 V q_e / (3m_e r_0 v_z c). \quad (9)$$

dz is $v_z dt$, so the equation can also be written as

$$dt = -L^2 V q_e / (3m_e r_0 v_z^2 c). \quad (10)$$

When the voltage on the exterior of a Faraday cage is sinusoidal it will result in modulation of the transit time for an electron beam. For $v_z = 0.1 c$, $r_0 = L = 0.5$ m, and 100 volts on the surface of the cage Eq. 10 evaluates to 11 ps. At a frequency of 10 MHz that results in a phase modulation of 0.04 degree peak. Phase coherent detection methods are extremely sensitive, and the magnitude of the effect is well within the range that is experimentally accessible.

The vector potential associated with the current in the wire leading to the sphere was neglected in the calculations, so the solution is only valid for quasi-static configurations. That requires that the vacuum wavelength of the excitation be much longer than the dimensions of the apparatus. When driving a capacitive load the current in the wire will be proportional to frequency, which should make it easy to experimentally identify any contribution from the vector potential.

Some insight into the consequences of the singularity at $v_z = 0$ can be gained by considering the case where the sphere is moving in the laboratory frame of reference and the particle is moving slowly relative to the sphere. In that case the sphere moves a large distance during in the time that the particle takes to traverse the diameter of the sphere. The distance traveled by the sphere goes to ∞ as the differential particle velocity goes to zero. The solution for a stationary sphere can be viewed as a limiting case of the more general solution. The singularity at $v_z = 0$ might be removable when the sphere is

retarded in the laboratory frame of reference and the stationary solution obtained by a limiting process, in which case the solution obtained overestimates the magnitude of the effect at low particle velocities.

In a mathematical sense the V in the equations should be written as dV , because the voltage behaves like an infinitesimal value. Quadratic terms appear when either dz or dV is large.

As is shown in the SOM, the v_z^2 terms drop out of the solution for the first derivative in the second frame of reference. The v_z^3 terms do not drop out, but their contribution to Eq. 9 is very small.

The acceleration of the particle will not be parallel to the velocity vector when there is a magnetic field inside the sphere. The Thomas precession^{1,2} will need consideration in those solutions.

III. THE AHARONOV-BOHM EFFECT

The Compton wavelength for an electron is $h/(m_e c)$. When expressed in units of the Compton wavelength Eq. 9 becomes

$$d\phi = -L^2 V q_e / (3hr_0 v_z). \quad (11)$$

After substituting $L^2 = 2r_0 L$ and $h = 2\pi\hbar$ the equation becomes

$$d\phi = -LV q_e / (3\pi\hbar v_z).$$

Finally, substituting $L = v_z t_L$,

$$d\phi = -\frac{1}{3\pi} V q_e t_L / \hbar.$$

The coefficient in the scalar Aharonov-Bohm effect⁴ is -1 rather than the $-\frac{1}{3\pi}$ of this solution. The discrepancy is being investigated. There are indications that its basis is the neglect of the odd derivatives in the calculation.

The time t_L in this solution cannot be replaced by the time t , as that would imply that the phase increases linearly with time. As Eq. 11 shows, the phase is quadratic in L , and therefore quadratic in t , when the path length is less than the diameter of the sphere.

¹J. Aharoni, *The Special Theory of Relativity*, (Oxford Press, London, 1965)

²J. D. Jackson, *Classical Electrodynamics*, (John Wiley and Sons, New York, 1975)

³Morse, P. M., & Feshbach, H. 1953, *Methods of Theoretical Physics*, Vol 1 (McGraw-Hill, New York)

⁴(https://en.wikipedia.org/wiki/Aharonov%E2%80%93Bohm_effect)